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# Front dynamics for an anisotropic reaction–diffusion equation

Sergei Fedotov†
Department of Mathematics, UMIST, Manchester M60 1QD, UK

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**Abstract.** The effects of anisotropic diffusion with a finite velocity on propagating fronts in a reaction–diffusion equation are examined within the framework of Hamilton–Jacobi theory. It is found that in the long-time large-distance asymptotic limit the Hamiltonian dynamical system associated with the reaction–diffusion equation has a structure identical to that of general relativity. It is shown that the function which determines the position of the reaction front and its speed can be interpreted as the action functional for a relativistic particle moving in both gravitational and electromagnetic fields. The diffusivity tensor determines the metric tensor of the four-dimensional Riemannian space of general relativity, while the speed of light corresponds to the finite speed of diffusion waves. The mass of the relativistic particle and scalar potential are found to be functions of the reaction rate coefficient and relaxation time. The analogy with general relativity theory allows us to find an explicit formula for the reaction front position.

## 1. Introduction

The last decade has seen increasingly detailed development of the theory of travelling waves in reaction–diffusion systems [1–6]. Due to its relative simplicity, special attention has been paid to the Fisher–Kolmogorov–Petrovskii–Piskunov (FKPP) equation, describing front propagation into an unstable state. The major feature of this process is that the dynamics of the reaction front are determined by the processes taking place at the leading edge of the front profile. However, in most cases the transport process is described by a diffusion approximation. As a result, the rate at which the wave propagates throughout the reaction–diffusion system can be overestimated. Physical reasoning supports this observation because the density field predicted by the diffusion approximation has higher tails than the density of the real transport process [7–9]. To deal with this problem several workers have introduced the hyperbolic correction to the diffusion approximation, taking into account the finite speed of the transport process [10–18]. An alternative way to overcome this problem is to introduce a cut-off for the nonlinear reaction term in such a way that the reaction rate becomes zero when the scalar field is less than the small parameter $\varepsilon$ [19, 20]. Such a regularization procedure reduces the propagation speed and results in a shift with the logarithmic form $K(\ln \varepsilon)^{-2}$.

In previous papers [15, 16], we have presented a formulation of reaction front dynamics in terms of special relativity theory, where the diffusion wave speed plays the role of the speed of light. We have found that in the long-time large-distance asymptotic limit the Hamiltonian dynamical system associated with the reaction–diffusion equation is similar to that of classical relativistic mechanics. This analogy is rooted in the fact that, in both cases, there is a finite propagation speed. In the solution of the generalized FKPP equation, the scalar field $\rho$ has

† Web-page: http://www.ma.umist.ac.uk/sf/index.html

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the asymptotic form $\rho \sim \exp(G/\varepsilon)$, where $\varepsilon$ is the small parameter describing the long-time large-distance limit and $G$ obeys the relativistic Hamilton–Jacobi equation defining the Hamiltonian dynamical system associated with the reaction–diffusion equation. One of the main advantages of this approach over a conventional analysis of travelling wave solutions is that the front dynamics can be described by a first-order partial differential equation rather than a second-order equation [21, 22]. It should be mentioned that there is another kinematic approach for waves in reaction–diffusion systems based on the curvature kinematics (see, for example, the review paper [23] and references therein). It is particular useful for a description of the spiral wave dynamics in the two-dimensional case. However, one cannot use this approach for the analysis of the front-jump phenomenon [5, 6, 16].

It is quite natural to ask whether or not there exists a *general relativity analogy* and if so how both gravitational and electromagnetic fields associated with a reaction–diffusion equation can be determined. It is the purpose of this paper to find the answer to this question and show that the function which determines the reaction front position can be derived from a variational principle of general relativity theory. The central result of this paper is that for the anisotropic reaction–diffusion equation the reaction front position can be found exactly from the general relativity Hamilton–Jacobi equation [23]. The diffusivity tensor determines the metric tensor of the four-dimensional Riemannian space of general relativity and the finite speed of diffusion waves can be regarded as the speed of light.

The theory presented in this paper is an approximate one analogous to the WKB approximation and the relation of geometric optics to electrodynamics. The mathematical basis of such a theory is an asymptotic expansion in powers of a small parameter $\varepsilon$ that is the ratio of a characteristic width of the travelling wave profile to the typical length scale of the problem. The basic equation governing the reaction front dynamics is for the lowest-order terms. Due to this approximation, the theory cannot describe the phenomena involving the interaction of the reaction fronts, the appearance of singularities, etc. It should be noted that here we consider only the special class of reaction terms of FKPP type. The method presented is valid only for the analysis of the propagation into an unstable state and cannot be used for the trigger waves propagating into a metastable state [20].

## 2. Reaction front propagation for the anisotropic reaction–diffusion equation

Our analysis of reaction front propagation in anisotropic media begins with the transport equation for a scalar field $\rho(t, \boldsymbol{r})$,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \boldsymbol{J} = U(\varepsilon \boldsymbol{r})\rho f(\rho) \qquad \boldsymbol{r} = (x^1, x^2, x^3) \tag{1}$$

where $\boldsymbol{J}$ is the mass flux and the nonlinear source term on the right-hand side of (1) is of KPP type [5, 6, 16]; that is

$$\max_{\rho \in [0,1]} f(\rho) = f(0) = 1 \qquad f(1) = 0. \tag{2}$$

The function $f(\rho) = 1 - \rho$ corresponds to the logistic growth [1–4]. The reaction rate parameter $U(\varepsilon \boldsymbol{r})$ is assumed to be a slowly varying function of the space coordinate $\boldsymbol{r}$; $\varepsilon$ is a small parameter. The reason for taking this dependence into account is that it might induce the phenomenon of a wavefront jump [5, 6].

It is well known that in an anisotropic medium, the direction of the mass flux $\boldsymbol{J}$ is, in general, not that of the gradient of the scalar field $\nabla \rho$. The component of the mass flux vector

$\boldsymbol{J}(t, \boldsymbol{r})$ can be written as

$$J^i(t, \boldsymbol{r}) = -\frac{1}{2}\sum_{k=1}^{3} D^{ik}\frac{\partial \rho}{\partial x^k} \tag{3}$$

where $D^{ik}$ is the diffusivity tensor. However, this classical approach to the transport process, in which one expresses an instantaneous dependence of the flux on the gradient, is not sufficiently accurate and may result in an overestimation of the speed of propagating fronts [9–18]. In order to investigate the role of inertia effects in the anisotropic transport process described by equation (1), we introduce the relaxation time $\tau$ in a such way that the component of the flux $\boldsymbol{J}$ can be determined by Cattaneo's law [8]

$$J^i(t, \boldsymbol{r}) = -\frac{1}{2\tau}\sum_{k=1}^{3}\int_0^t \exp\left(-\frac{t-s}{\tau}\right) D^{ik}(\varepsilon \boldsymbol{r})\frac{\partial \rho}{\partial x^k}(s, \boldsymbol{r})\mathrm{d}s. \tag{4}$$

Here the matrix $D^{ik}$ is assumed to be symmetric and positive definite. If inertia is neglected ($\tau = 0$), then we have Fick's law (3) and equation (1) together with (3) can be written in the form of the classical FKPP equation [1–6].

Equations (1) and (4) can be rewritten as a single equation for $\rho$

$$\frac{\partial \rho}{\partial t} = \frac{1}{2\tau}\sum_{i,k=1}^{3}\int_0^t \exp\left(-\frac{t-s}{\tau}\right)\frac{\partial}{\partial x^i} D^{ik}(\varepsilon \boldsymbol{r})\frac{\partial \rho}{\partial x^k}(s, \boldsymbol{r})\mathrm{d}s + U(\varepsilon \boldsymbol{r})\rho f(\rho). \tag{5}$$

This equation incorporates the combined effects of anisotropic diffusion with finite velocity, exponential growth and nonlinear saturation. It should be noted that the initial flux $J^i(0, \boldsymbol{r})$ is assumed to be zero.

To analyse the reaction front dynamics corresponding to (5), the initial distribution for $\rho$ has to be specified. It is well known that the propagation rate may vary from the minimum velocity value to infinity depending on the initial condition [1–6]. Here we assume the initial distribution to be in the form of the indicator function $\chi_{S_0}$ of the set $S_0$

$$\rho(0, \boldsymbol{r}) = \chi_{S_0} = \begin{cases} 1 & \text{if} \quad \boldsymbol{r} \in S_0 \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

This initial condition ensures that the reaction front propagates at the minimum velocity. To avoid unnecessary complications, $S_0$ is assumed to be the convex set. For example, the set $S_0$ can be a ball of radius $R/\varepsilon$, such that

$$\rho(0, \boldsymbol{r}) = \begin{cases} 1 & \text{if} \quad (x^1)^2 + (x^2)^2 + (x^3)^2 \leqslant \dfrac{R^2}{\varepsilon^2} \qquad \varepsilon \ll 1. \\ 0 & \text{otherwise} \end{cases}$$

One can see from here that the initial distribution involves a small parameter $\varepsilon$ that plays a very important role in what follows.

It is well known that the hyperbolic scaling procedure $t \to t/\varepsilon$, $\boldsymbol{r} \to \boldsymbol{r}/\varepsilon$ yields the large-scale geometric front propagation for the FKPP equation [5, 6]. The behaviour of the rescaled field $\rho^\varepsilon(t, \boldsymbol{r}) = \rho(t/\varepsilon, \boldsymbol{r}/\varepsilon)$ may be explained in terms of a simple geometric picture. Since the nonlinear function $\rho^\varepsilon f(\rho^\varepsilon)$ on the right-hand side of (5) is equal to zero only if $\rho^\varepsilon = 0$ and $\rho^\varepsilon = 1$, we may argue that in the limit $\varepsilon \to 0$ the solution $\rho^\varepsilon$ converges to the indicator function of the set $S_t$ [21, 22]

$$\lim_{\varepsilon \to 0} \rho^\varepsilon(t, \boldsymbol{r}) = \chi_{S_t} = \begin{cases} 1 & \text{if} \quad \boldsymbol{r} \in S_t \\ 0 & \text{otherwise.} \end{cases} \tag{7}$$

The boundary of the set $S_t$ can be regarded as a reaction front describing the interface dynamics between the stable ($\rho^\varepsilon = 1$) and unstable ($\rho^\varepsilon = 0$) phases.

After hyperbolic scaling $t \to t/\varepsilon$, $\boldsymbol{r} \to \boldsymbol{r}/\varepsilon$, equation (5) can be rewritten as (see the appendix)

$$\varepsilon\tau\frac{\partial^2\rho^\varepsilon}{\partial t^2} + \left(1 - \tau U(\boldsymbol{r}) f(\rho^\varepsilon) - \tau U(\boldsymbol{r})\rho^\varepsilon \frac{\mathrm{d}f(\rho^\varepsilon)}{\mathrm{d}\rho^\varepsilon}\right)\frac{\partial\rho^\varepsilon}{\partial t}$$
$$= \frac{\varepsilon}{2}\sum_{i,k=1}^{3}\frac{\partial}{\partial x^i}D^{ik}(\boldsymbol{r})\frac{\partial\rho^\varepsilon}{\partial x^k} + \frac{U(\boldsymbol{r})}{\varepsilon}\rho^\varepsilon f(\rho^\varepsilon) \tag{8}$$

while the initial condition has the form

$$\rho^\varepsilon(0,\boldsymbol{r}) = \begin{cases} 1 & \text{if} \quad \boldsymbol{r} \in S_0 \\ 0 & \text{otherwise.} \end{cases} \tag{9}$$

Now we turn to the problem of finding the front dynamics for (8) and (9) in the limit $\varepsilon \to 0$.

## 3. Geometric optic approximation

Here we present a heuristic derivation of the Hamilton–Jacobi equation describing reaction front dynamics. Let us write down $\rho^\varepsilon(t,\boldsymbol{r})$ in exponential form

$$\rho^\varepsilon(t,\boldsymbol{r}) = \exp\left(-\frac{G^\varepsilon(t,\boldsymbol{r})}{\varepsilon}\right) \qquad G^\varepsilon(t,\boldsymbol{r}) \geqslant 0 \tag{10}$$

where the non-negative function $G^\varepsilon$ describing the logarithmic asymptotic of the concentration field plays a very important role. It follows from (10) that as long as the function $G(t,\boldsymbol{r}) = \lim_{\epsilon\to 0} G^\varepsilon(t,\boldsymbol{r})$ is positive ($G(t,\boldsymbol{r}) > 0$), the rescaled field $\rho^\varepsilon(t,\boldsymbol{r}) \to 0$ as $\varepsilon \to 0$. So the boundary of the set $S_t$ (see (7)), described above as the reaction front position, is nothing else but the boundary of the set where $G(t,\boldsymbol{r}) > 0$. Therefore, we may argue that the reaction front position can be determined as

$$\delta S_t = \{\boldsymbol{r} \in R^3 : G(t,\boldsymbol{r}) = 0\}.$$

Now we are in a position to determine the function $G(t,\boldsymbol{r})$. First, let us find an equation for $G^\varepsilon(t,\boldsymbol{r})$. Inserting (10) into (8), we find that $G^\varepsilon(t,\boldsymbol{r})$ satisfies the nonlinear PDE

$$\tau\left(\frac{\partial G^\varepsilon}{\partial t}\right)^2 - (1 - \tau U(\boldsymbol{r}) f)\frac{\partial G^\varepsilon}{\partial t} - \frac{1}{2}\sum_{i,k=1}^{3} D^{ik}(\boldsymbol{r})\frac{\partial G^\varepsilon}{\partial x^i}\frac{\partial G^\varepsilon}{\partial x^k} - U(\boldsymbol{r}) f\left(\exp\left[-\frac{G^\varepsilon(t,\boldsymbol{r})}{\varepsilon}\right]\right)$$
$$= \varepsilon\left[\tau\frac{\partial^2 G^\varepsilon}{\partial t^2} - \frac{1}{2}\sum_{i,k=1}^{3}\left(D^{ik}(\boldsymbol{r})\frac{\partial^2 G^\varepsilon}{\partial x^i \partial x^k} + \frac{\partial G^\varepsilon}{\partial x^k}\frac{\partial D^{ik}(\boldsymbol{r})}{\partial x^i}\right)\right]$$
$$-U(\boldsymbol{r})\tau\frac{\partial G^\varepsilon}{\partial t}\frac{\mathrm{d}f}{\mathrm{d}\rho^\varepsilon}\exp\left[-\frac{G^\varepsilon(t,\boldsymbol{r})}{\varepsilon}\right]. \tag{11}$$

Since

$$\lim_{\epsilon\to 0} f\left(\exp\left[-\frac{G^\varepsilon(t,\boldsymbol{r})}{\varepsilon}\right]\right) = 1 \qquad \lim_{\epsilon\to 0}\exp\left(-\frac{G^\varepsilon}{\varepsilon}\right) = 0 \tag{12}$$

provided $G^\varepsilon(t,\boldsymbol{r}) > 0$, it follows from (11) that the limiting function

$$G(t,\boldsymbol{r}) = -\lim_{\epsilon\to 0}\varepsilon\ln\rho^\varepsilon(t,\boldsymbol{r}) \tag{13}$$

obeys the nonlinear PDE of the first order

$$\left(\frac{\partial G}{\partial t}\right)^2 - \left(\frac{1}{\tau} - U(\boldsymbol{r})\right)\frac{\partial G}{\partial t} - \frac{1}{2\tau}\sum_{i,k=1}^{3} D^{ik}(\boldsymbol{r})\frac{\partial G}{\partial x^i}\frac{\partial G}{\partial x^k} - \frac{U(\boldsymbol{r})}{\tau} = 0 \qquad (14)$$

provided

$$G(t, \boldsymbol{r}) > 0.$$

If $\tau = 0$, then

$$\frac{\partial G}{\partial t} + \frac{1}{2}\sum_{i,k=1}^{3} D^{ik}(\boldsymbol{r})\frac{\partial G}{\partial x^i}\frac{\partial G}{\partial x^k} + U(\boldsymbol{r}) = 0. \qquad (15)$$

Freidlin was the first to show that in the long-time large-distance limit the problem of the travelling wave solution to the classical FKPP equation can be reduced to that of the Hamilton–Jacobi equation (15) [5, 6]. The function $G(t, \boldsymbol{r})$ can be found from the variational problem

$$G(t, \boldsymbol{r}) = \min\{\int_0^t L\left(\boldsymbol{r}(s), \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}s}(s)\right)\mathrm{d}s : \boldsymbol{r}(0) \in S_0, \boldsymbol{r}(t) = \boldsymbol{r}\}$$

where $L$ is the Lagrangian function of classical mechanics

$$L = \frac{1}{2}\sum_{i,k=1}^{3} D_{ik}(\boldsymbol{r})\frac{\mathrm{d}x^i}{\mathrm{d}s}\frac{\mathrm{d}x^k}{\mathrm{d}s} - U(\boldsymbol{r}).$$

The reaction rate parameter $U(\boldsymbol{r})$ plays the role of potential energy, the matrix $D_{ik}(\boldsymbol{r}) = (D^{ik}(\boldsymbol{r}))^{-1}$ determines the positive-definite quadratic form of the kinetic energy and $x^i$ may be regarded as the generalized coordinates. It is quite remarkable that the concepts of classical mechanics lead to a new formulation of reaction front dynamics for the reaction–diffusion system.

Our problem is now to find a solution to equation (14) that can be considered a generalization of (15) and possibly to find a new interpretation of the phenomenological parameters $U(\boldsymbol{r})$, $D^{ik}(\boldsymbol{r})$ and $\tau$.

## 4. General relativity Hamilton–Jacobi equation

The interesting feature of equation (14) is that it can be rewritten in the form of the Hamilton–Jacobi equation for a relativistic charged particle ($e = 1$) in the presence of both gravitational and electromagnetic fields [24],

$$\sum_{\alpha,\beta=1}^{4} g^{\alpha\beta}\left(\frac{\partial G}{\partial z^\alpha} - \frac{1}{c}A_\alpha\right)\left(\frac{\partial G}{\partial z^\beta} - \frac{1}{c}A_\beta\right) + m^2c^2 = 0 \qquad (16)$$

where the new four-dimensional radius vector $z^\alpha$ is determined as

$$z^0 = ct \qquad z^i = x^i \qquad i = 1, 2, 3. \qquad (17)$$

Here we have introduced the 4-potential of the electromagnetic field $A_\alpha$ such that the space components of $A_\alpha$ ($\alpha = 1, 2, 3$) forming the vector potential of the field are zero

$$A_1 = A_2 = A_3 = 0 \qquad (18)$$

and the time component $A_0$ forming the scalar potential is

$$A_0 = -\varphi(\boldsymbol{r}) \qquad \varphi(\boldsymbol{r}) = \frac{1}{2}\left(U(\boldsymbol{r}) - \frac{1}{\tau}\right). \tag{19}$$

The contravariant metric tensor $g^{\alpha\beta}$ in (16) has the following form:

$$g^{\alpha\beta} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & d^{11} & d^{12} & d^{13} \\ 0 & d^{21} & d^{22} & d^{23} \\ 0 & d^{31} & d^{32} & d^{33} \end{pmatrix} \tag{20}$$

where the contravariant tensor $d^{ik}$ is determined as follows:

$$d^{ik}(\boldsymbol{r}) = \frac{D^{ik}(\boldsymbol{r})}{\max_{i,k,r} D^{ik}(\boldsymbol{r})}. \tag{21}$$

The mass $m(\boldsymbol{r})$ and 'speed of light' $c$ are

$$m(\boldsymbol{r}) = \frac{1}{2c^2}\left(U(\boldsymbol{r}) + \frac{1}{\tau}\right) \qquad c^2 = \frac{\max_{i,k,r} D^{ik}(\boldsymbol{r})}{2\tau}. \tag{22}$$

The result that equation (14), which governs the dynamics of the reaction front, can be rewritten as the general relativity Hamilton–Jacobi equation is of basic importance for us. The reason for this is that it allows us to write down the solution of (14) as in [24],

$$G = \min\left\{-mc\int \mathrm{d}s + \frac{1}{c}\sum_{\alpha=1}^{4}\int A_\alpha \,\mathrm{d}z^\alpha\right\} \tag{23}$$

where $\mathrm{d}s$ is the line element of the four-dimensional Riemannian space of general relativity

$$-(\mathrm{d}s)^2 = \sum_{\alpha,\beta=1}^{4} g_{\alpha\beta}\,\mathrm{d}z^\alpha\,\mathrm{d}z^\beta \tag{24}$$

and $g_{\alpha\beta}$ is the covariant metric tensor

$$g_{\alpha\beta} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & d_{11} & d_{12} & d_{13} \\ 0 & d_{21} & d_{22} & d_{23} \\ 0 & d_{31} & d_{32} & d_{33} \end{pmatrix}. \tag{25}$$

Here the covariant tensor $d_{ik}$ can be determined from

$$\sum_{i=1}^{3} d_{ki}d^{ij} = \delta_k^j \tag{26}$$

where $\delta_k^j$ is the familiar Kronecker delta.

The explicit solution (23) can be rewritten in terms of the Lagrangian function $L$

$$G(t,\boldsymbol{r}) = \min\left\{\int_0^t L\left(\boldsymbol{r}(s), \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}s}(s)\right)\mathrm{d}s : \boldsymbol{r}(0) \in S_0, \boldsymbol{r}(t) = \boldsymbol{r}\right\} \tag{27}$$

where

$$L = -m(\boldsymbol{r})c^2\sqrt{1 - \frac{1}{c^2}\sum_{i,k=1}^{3} d_{ik}(\boldsymbol{r})\frac{\mathrm{d}x^i}{\mathrm{d}s}\frac{\mathrm{d}x^i}{\mathrm{d}s}} - \varphi(\boldsymbol{r}). \tag{28}$$

In terms of the phenomenological parameters $U(\boldsymbol{r})$, $D^{ik}(\boldsymbol{r})$ and $\tau$ the Lagrangian function $L$ takes the form

$$L = -\frac{1}{2}\left(U(\boldsymbol{r}) + \frac{1}{\tau}\right)\sqrt{1 - \frac{2\tau}{\max_{i,k,\boldsymbol{r}} D^{ik}(\boldsymbol{r})}\sum_{i,k=1}^{3} d_{ik}(\boldsymbol{r})\frac{\mathrm{d}x^i}{\mathrm{d}s}\frac{\mathrm{d}x^i}{\mathrm{d}s}} - \frac{1}{2}\left(U(\boldsymbol{r}) - \frac{1}{\tau}\right). \tag{29}$$

Thus the expression (27) provides an explicit solution to the reaction position problem for the generalized FKPP equation (5) with the initial condition (6). The exact formula for the reaction front position and its propagation rate can be obtained when the reaction rate $U$ and the diffusion tensor $D^{ik}$ are constant.

## 5. Explicit formula for the reaction front position

Let us denote by $H(\boldsymbol{r}, \boldsymbol{p})$ the Hamiltonian function associated with the variational problem (27), where $\boldsymbol{p}$ is a generalized momentum. By using the Legendre transformation

$$H(\boldsymbol{r}, \boldsymbol{p}) = \max_{\boldsymbol{k}}(\boldsymbol{p}\cdot\boldsymbol{k} - L(\boldsymbol{r}, \boldsymbol{k})) \tag{30}$$

we can find

$$H(\boldsymbol{r}, \boldsymbol{p}) = \sqrt{m^2(\boldsymbol{r})c^4 + c^2\sum_{i,k=1}^{3} d^{ik}(\boldsymbol{r})p_i p_k} + \varphi(\boldsymbol{r}). \tag{31}$$

The optimal trajectories giving the minimum to the functional (27) satisfy the Hamilton equations

$$\frac{\mathrm{d}x^i}{\mathrm{d}s} = \frac{\partial H}{\partial p_i} \qquad \frac{\mathrm{d}p_i}{\mathrm{d}s} = -\frac{\partial H}{\partial x^i}. \tag{32}$$

When the parameters $m^2(\boldsymbol{r})$, $d^{ik}(\boldsymbol{r})$ and $\varphi(\boldsymbol{r})$ are independent from the space coordinate $\boldsymbol{r}$, we have

$$\frac{\mathrm{d}x^i}{\mathrm{d}s} = \text{constant} \qquad p_i = \text{constant}. \tag{33}$$

It follows from this that the optimal trajectories are straight lines. Taking into account the boundary conditions in (27) we can find

$$x^i_{opt}(s) = \frac{x^i - y^i}{t}s + y^i \qquad \boldsymbol{y} \in \delta S_0 \tag{34}$$

where $\delta S_0$ is a border of the set $S_0$; that is, the initial reaction front. When these trajectories are substituted into (27) we obtain

$$G(t, \boldsymbol{r}) = \min_{\boldsymbol{y}\in S_0}\left\{-mc^2 t\sqrt{1 - \frac{1}{c^2t^2}\sum_{i,k=1}^{3} d_{ik}(x_i - y_i)(x_k - y_k)} - \varphi t\right\}. \tag{35}$$

Let us denote by $l^2_{\min}(\boldsymbol{r}, S_0)$ the following expression:

$$l^2_{\min}(\boldsymbol{r}, S_0) = \min_{\boldsymbol{y}\in S_0}\sum_{i,k=1}^{3} d_{ik}(x_i - y_i)(x_k - y_k). \tag{36}$$

Then

$$G(t,\boldsymbol{r}) = -mc^2 t\sqrt{1-\left(\frac{l_{\min}(\boldsymbol{r},S_0)}{ct}\right)^2} - \varphi t. \tag{37}$$

It is clear from (37) that the theory is valid as long as

$$1-\left(\frac{l_{\min}(\boldsymbol{r},S_0)}{ct}\right)^2 \geqslant 0.$$

This condition has a very simple physical interpretation: relativity theory forbids the particle from propagating at a speed which exceeds the velocity of light $c$. As a result, an inequality $l_{\min}(\boldsymbol{r},S_0) \leqslant ct$ must hold. It follows from (37) that the reaction front position $\delta S_t = \{\boldsymbol{r} \in R^3 : G(t,\boldsymbol{r}) = 0\}$ at time $t$ can be found from

$$-mc^2\sqrt{1-\left(\frac{l_{\min}(\boldsymbol{r},S_0)}{ct}\right)^2} - \varphi = 0. \tag{37$'$}$$

Taking into account the phenomenological expressions for $m$ and $\varphi$ (see (19) and (22)), we can find the reaction front position as follows:

$$\delta S_t = \{\boldsymbol{r} \in R^3 : l_{\min}(\boldsymbol{r},S_0) = ut\} \tag{38}$$

where

$$u = c\sqrt{1-\left(\frac{1-\tau U}{1+\tau U}\right)^2} = \frac{\sqrt{2\max_{i,k} D^{ik}U}}{1+\tau U} \qquad \tau U \leqslant 1. \tag{39}$$

We can also give the asymptotic behaviour of the scalar field $\rho^\varepsilon(t,\boldsymbol{r})$ in terms of $l_{\min}(\boldsymbol{r},S_0)$

$$\lim_{\varepsilon\to 0}\rho^\varepsilon(t,\boldsymbol{r}) = \begin{cases} 1 & \text{if} \quad l_{\min}(\boldsymbol{r},S_0) > ut \\ 0 & \text{otherwise.} \end{cases} \tag{40}$$

If the initial set $S_0$ is a ball with radius $R$, and $D^{ik} = D\delta^{ik}$, then $l_{\min}(\boldsymbol{r},S_0) = R - r$ and $u = \sqrt{2DU}(1+\tau U)^{-1}$.

## 6. Summary

Basically, we have extended the classical treatment of the FKPP equation due to Freidlin [5, 6] to include the phenomenon of anisotropic diffusion with a finite velocity. We have shown that in the long-time large-distance asymptotic limit the Hamiltonian dynamical system associated with the anisotropic reaction–diffusion equation has a structure identical to that of general relativity theory. The function determining the position of the reaction front and its speed is nothing else but the action functional for a particle in both gravitational and electromagnetic fields. The metric tensor of the four-dimensional Riemannian space of general relativity has been determined through the diffusivity tensor, while the speed of light corresponds to the finite speed of diffusion waves. The mass of the relativistic particle and scalar potential have been found to be functions of the reaction rate coefficient and relaxation time. For constant values of the reaction rate function and the diffusivity tensor, the analogy with general relativity theory has allowed us to find the explicit formula for the reaction front position and its speed.

It is well known that in the so-called weak-noise limit, the function $G$ may play the role of a thermodynamical potential for the dissipative system described by the Fokker–Planck equation [25]. It would be very interesting to find a broad class of dissipative systems for which such a

Lyapunov function corresponds to the action functional occurring in general relativity theory. It would also be interesting to consider the 'relativistic effects' induced by the finite speed of transport processes in terms of variational inequalities and viscosity solutions [26, 27].

An important application of the result of this paper may be the propagation of a reaction front in a turbulent combustion flow [28]. It is well known that the macroscale equations for turbulent heat/mass transport involve effective anisotropic transport processes with a finite velocity [29]. It is also of great interest to analyse the reaction front dynamics in a slowly varying medium when the the reaction front jump phenomenon might happen [5, 6, 16].

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## Appendix

The system (1) and (4) can be rewritten as follows:

$$\frac{\partial \rho}{\partial t} + \sum_{i=1}^{3} \frac{\partial J^i}{\partial x^i} = U(\varepsilon \boldsymbol{r}) \rho f(\rho) \qquad \boldsymbol{r} \in R^3 \tag{A1}$$

$$\frac{\partial J^i}{\partial t} = -\frac{J^i}{\tau} - \frac{1}{\tau} \sum_{k=1}^{3} D^{ik}(\varepsilon \boldsymbol{r}) \frac{\partial \rho}{\partial x^k}. \tag{A2}$$

It is easy to see that an expression for $J^i$ in (4) is just a solution of the differential equation (A2) under the initial condition $J^i(0, \boldsymbol{r}) = 0$.

By differentiating the first equation with respect to time $t$ and the second one with respect to the space coordinate $x^i$ we obtain

$$\frac{\partial^2 \rho}{\partial t^2} + \sum_{i=1}^{3} \frac{\partial^2 J^i}{\partial t \partial x^i} = U(\varepsilon \boldsymbol{r}) \left( f(\rho) + \rho \frac{\mathrm{d} f(\rho)}{\mathrm{d} \rho} \right) \frac{\partial \rho}{\partial t} \qquad \boldsymbol{r} \in R^3 \tag{A3}$$

$$\frac{\partial^2 J^i}{\partial x^i \partial t} = -\frac{1}{\tau} \frac{\partial J^i}{\partial x^i} - \frac{1}{\tau} \sum_{k=1}^{3} \frac{\partial}{\partial x^i} D^{ik}(\varepsilon \boldsymbol{r}) \frac{\partial \rho}{\partial x^k}. \tag{A4}$$

By using (A1) the last equation can be rewritten as

$$\sum_{i=1}^{3} \frac{\partial^2 J^i}{\partial x^i \partial t} = -\frac{1}{\tau} \left( -\frac{\partial \rho}{\partial t} + U(\varepsilon \boldsymbol{r}) \rho f(\rho) \right) - \frac{1}{\tau} \sum_{i,k=1}^{3} \frac{\partial}{\partial x^i} D^{ik}(\varepsilon \boldsymbol{r}) \frac{\partial \rho}{\partial x^k}. \tag{A5}$$

Substitution of the expression for $\sum_{i=1}^{3} \frac{\partial^2 J^i}{\partial x^i \partial t}$ given by (A5) into (A3) and multiplication by $\tau$ give

$$\tau \frac{\partial^2 \rho}{\partial t^2} + \left( 1 - \tau U(\varepsilon \boldsymbol{r}) f(\rho) - \tau U(\varepsilon \boldsymbol{r}) \rho \frac{\mathrm{d} f(\rho)}{\mathrm{d} \rho} \right) \frac{\partial \rho}{\partial t}$$
$$= \sum_{i,k=1}^{3} \frac{\partial}{\partial x^i} D^{ik}(\varepsilon \boldsymbol{r}) \frac{\partial \rho}{\partial x^k} + U(\varepsilon \boldsymbol{r}) \rho f(\rho).$$

After hyperbolic scaling $t \to t/\varepsilon$, $\boldsymbol{r} \to \boldsymbol{r}/\varepsilon$, this equation can be rewritten as (8).

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